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THE STABILITY OF ROTATION OF A SATELLITE RING*

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The stability of steady rotation of a closed flexible filament, closed in a ring around an attracting centre is investigated as a function of the filament extensibility. Characteristics ensuring the stability of rotation of the ring are determined. These do not correspond to Hooke's law. The results are used to investigate the stability of steady rotation of a large number of artificial satellites connected in series along the orbit in a ring by weightless cables of variable length. A wide class of control laws is indicated for the tension of the connecting cables that ensure the stability of the satellite ring. An analogy in the dynamics of meteor and satellite rings is described a model of a meteor ring formed by the equivalent flexible ring having a defined law of extensibility is proposed.

The problem of the motion of a ring round an atrracting centre was first formulated by Laplace /1/ in connection with the observation of Saturn's rings. Laplace demonstrated the instability of its rotation on a model of absolutely rigid homogeneous ring, and he determined the equilibrium form of its cross section using the model of a ring of incompressible fluid. Subsequently Kovalevskaya /2/ studied the form of a liquid ring. Following Laplace, Maxwell showed /3/ that rotation of rigid rings with an inhomogeneous mass distribution is, as a rule, unstable. The rotation of rings of incompressible fluid also proved to be unstable. As a possible model of a stable ring Maxwell proposed a system of large numbers of material points of the same mass. In steady motion all points dispose themselves at the vertices of a regular polygon whose centre is the attracting centre. The stationary rotation of a ring of N points of common mass m is stable to a first approximation when the following condition /3/ is satisfied:

$$m < 2{,}298MN^{-2} \tag{0.1}$$

where M is the mass of the central body.

Recently, interest in continuous rings (in the sense of Laplace's initial assumptions) has been revived. It is assumed that electric power space stations, enterprises, and human settlement may be connected in rings around the Earth /4/, for which a flexible filament, linearly stretchable, is taken as the model of an artificial ring. The filament is in the form of a closed circle, whose rotation is unstable for any modulus of elasticity of the filament.** (**Independently of /4/ an analytic proof of instability was given by A.I. Morozov and A.M. Fridman who drew the attention of the authors to this problem.) The analysis of the motion of elastic rings to some extend complements the investigation of Laplace and Maxwell. The motion of a solid ring of present-day materials cannot be well approximated by the motion of a rigid ring /5/. The instability of uncontrolled elastic rings indicates the need for control. A very complex version of a stabilizing control was proposed in /4/. Below an alternative version, constructed using the example of natural rings is considered. It is distinguished by the simplicity of its construction.

1. Consider a homogeneous flexible and extensible filament in the form of a closed ring located in a field of a stationary attracting centre O to which is attached the inertial system of coordinates $OXYZ$. The usual equations of dynamics of a filament are /6/

$$\rho\frac{\partial^2\mathbf{r}}{\partial t^2}=\frac{\partial}{\partial s}(T\boldsymbol{\tau})-\rho\mu\mathbf{r}r^{-3} \tag{1.1}$$

where $\mathbf{r}(s,t)$ is the radius vector of a point of the filament at the instant of time t, s is a natural parameter measured along the filament in some initial state (not necessarily stressed), $0\leqslant s\leqslant l$, l is the initial length of the filament, T is the tension of the filament, $\boldsymbol{\tau}=(\partial\mathbf{r}/\partial s)\,|\partial\mathbf{r}/\partial s|^{-1}$ is the unit vector of the tangent to the filament line of curvature, ρ is the mass per unit length of the filament, and μ is the gravitational constant of the attracting centre. Eq.(1.1) must be supplemented by the conditions of closure of the filament in a ring,

$$\mathbf{r}(0,t)=\mathbf{r}(l,t),\quad \frac{\partial\mathbf{r}}{\partial s}(0,t)=\frac{\partial\mathbf{r}}{\partial s}(l,t) \tag{1.2}$$

and the dependence of the tension T on the filament elongation γ

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$$T = T(\gamma), \quad \gamma = |\partial \mathbf{r}/\partial s| \tag{1.3}$$

Eqs.(1.1) and (1.2) admit of a uniform rotation of the filament in a stationary plane along a circle of constant radius with centre at the point O. This is the state of apparent rest /6/, to which there corresponds by virtue of (1.1) a constant tension along the filament.

$$T = T_* = \rho(\omega^2 R^2 - \mu R^{-1}) \tag{1.4}$$

where R is the radius of the circle, and ω is the angular velocity of rotation. We will assume that the filament is subjected only to tension $T_* > 0$ to which there corresponds $R > R_0$ where $R_0 = (\mu\omega^{-2})^{1/3}$ is the radius of a circular Keplerian orbit that corresponds to the angular velocity ω. The natural parameter s is measured at the initial state of apparent rest. We require (1.3) and (1.4) to be consistent when $\gamma = 1$: $T(1) = T_*$.

We shall formulate the problem of stability as follows: what should the relation between the filament tension and its elongation (1.3) be for the steady rotation of the filament to be stable? Note that according to the results in /4/ Hooke's law $T = T_* + E(\gamma - 1)$ leads to instability (in this formula E is the modulus of elasticity of the filament, and $E > T_*$ since $\gamma = 1 - T_* E^{-1} > 0$ must correspond to the unstressed state $T = 0$).

2. We introduce the system of coordinates $Oxyz$ which rotates at the angular velocity ω about the Oz axis perpendicular to the plane of steady rotation of the filament. In steady rotation the ring is stationary relative to the $Oxyz$ axes. The equations of small oscillations about the position of equilibrium with respect to the $Oxyz$ axes have the form

$$\begin{aligned} &b(u^{\cdot\cdot} - 2v^{\cdot} - 3u) = u'' - (2 + a)u - (1 + a)v' \\ &b(v^{\cdot\cdot} + 2u^{\cdot}) = (1 + a)u' + av'', \\ &b(w^{\cdot\cdot} + w) = w + w'' \\ &a = \frac{1}{T_*}\left(\frac{dT}{d\gamma}\right)_{\gamma=1}, \quad b = \frac{\rho\omega^2 R^2}{T_*} \end{aligned} \tag{2.1}$$

The dot and prime denotes differentiation with respect to the non-dimensional time $\tau = \omega t$ and the angle $\varphi = s/R$, which replaces the natural parameter s, and s, $0 \leqslant \varphi \leqslant 2\pi$, and $u(\varphi, \tau)$, $v(\varphi, \tau)$, $w(\varphi, \tau)$ are, respectively, the radial, transversal, and axial displacement of the point φ of the filament at instant τ relative to its position in steady motion. There are two dimensionless parameters a and b in Eq.(2.1). By virtue of (1.4) we have $T_* < \rho\omega^2 R^2$ and $b > 1$. The region $a > 1$ corresponds to Hooke's law $T = T_* + E(\gamma - 1)$, $E > T_*$ where we have instability /4/.

Let us investigate the region $1 > a > -\infty$.

Particular solutions of the linear equations with constant coefficients (2.1) are sought in the form $u = U\cos(\Omega\tau + \sigma\varphi)$, $v = V\sin(\Omega\tau + \sigma\varphi)$, $w = 0$ for oscillations in the Oxy plane, and in the form $u = v = 0$, $w = W\cos(\Omega\tau + \sigma\varphi)$ for oscillations in the Oz direction. The condition of closure of the filament into a ring (1.2), equivalent to a 2π periodicity with respect to φ, yields $\sigma = n$ an integer. For axial oscillations the characteristic equation

$$b\Omega^2 = b - 1 + n^2 \tag{2.2}$$

is independent of the control law (1.3), and by virtue of $b > 1$ has for all integral n two different real roots Ω. This shows that the ring is stable relative to axial displacements to a first approximation.

Let us consider the characteristic equation of oscillations in the plane of steady rotation of the ring

$$(b\Omega^2 + 3b - a - 2 - n^2)(b\Omega^2 - an^2) - (2b\Omega - (1 + a)n)^2 = 0 \tag{2.3}$$

If for all integral n Eq.(2.3) has four different real roots Ω, it means stability of the ring to a first approximation relative to displacements in the plane of steady rotation. If (2.3) has complex roots for some n, then such roots form complex conjugate pairs by virtue of the real coefficients in (2.3). In that situation we have an exponentially growing solution and the ring is unstable to a first approximation. When the sign of n changes the roots Ω also change sign, it is sufficient to consider only non-negative n.

When $n = 0$ we obtain $\Omega_{1,2} = 0$, $\Omega_{3,4} = \pm\sqrt{1 + (2 + a)/b}$. The solutions $u = c_1$, $v = c_2 + c_1\tau(2 + a - 3b)/(2b)$, $c_{1,2} = \mathrm{const}$, which define the transition to steady rotation that can be as close as desired to the initial, corresponds to zero roots. Such a transition does not affect the stability relative to radial displacements. The values of $\Omega_{3,4}$ are real and different, and the form $n = 0$ is steady when $a > -2 - b$. In the region $a < -2 - b$, which is denoted by crosses in Fig.1, the form $n = 0$ is unstable, and we have the possibility of increasing compression or widening of the circle, as a whole.

Consider the case when $n > 0$. Emergence from the stability region on its boundary is characterized by the appearance of a pair of multiple real roots Ω. Hence at the stability boundary the left hand side of (2.3) may be represented in the form $b^2(\Omega - p)^2(\Omega - q)(\Omega - r)$, where p, q, r are real numbers. This representation gives the equation of the stability boundary in parametric form

$$2p + q + r = 0,\ 3p^2 - qr = (2 + a + b + n^2 + an^2)/b, \qquad (2.4)$$
$$p^3 - pqr = 2n\,(1 + a)/b$$
$$p^2qr = n^2\,(an^2 - 1 - 3ab)/b^2$$

Using (2.4), the boundaries of stability region were numerically determined for $n \leqslant 20$ and for higher forms $n \gg 1$ asymptotic formulae were obtained which are in good agreement with numerical calculations, beginning at approximately $n = 10$. In Figs.1–3 the stability regions are shown shaded, and for convenience of representation the scales in regions $a < 0$ and $a > 0$ are different.

When $n = 1$ (Fig.1) the boundaries CB and EF become the curve $a = 1/b$ and the straight line $a = -0{,}037b$, as $b \to \infty$, and the coordinates of point E are $a = -1,\ b = 9$. For $n > 1$ the stability regions have a universal form, as for $n = 5$ in Fig.2. The boundary CB becomes the curve $a = 1/b$, as $b \to \infty$ and boundary CA becomes the straight line $b = (n^2 + 1)(3 - n^{-2})^{-1}$ as $a \to \infty$; the ordinate of point D when $n \gg 1$ is n^{-2} with an accuracy of within n^{-4}; the boundary

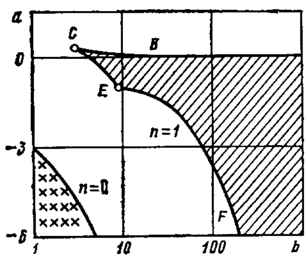

Fig.1

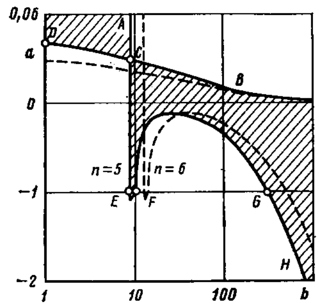

Fig.2

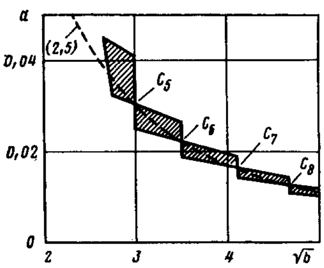

Fig.3

FGH is close to the curve $a = 4\sqrt{\beta\,(3\beta - 1)} + 1 - 7\beta,\ \beta = bn^{-2}$ when $n \gg 1$, point C has the coordinates $a = 3\,(4n^2 - 1)^{-1},\ b = (n^2 + 1 + 0{,}25n^{-2})\,(3 - 0{,}75n^{-2})^{-1}$, and points E, F, and G which lie on the line $a = -1$ have as abscissas $b_E = (n^2 + 1)/3,\ b_{F,G} = 6n^2 - 1 \mp 4n\sqrt{2n^2 - 1}$. On increasing n by unity, the stability region is displaced, as shown in Fig.2 by the dashed line for $n = 6$.

As can be seen in Figs.1 and 2, the nature of the stability and instability in regions $a \geqslant 1/b$ and $a \leqslant 0$ is entirely different. In regions $a \geqslant 1/b$ we always have an instability of the form $n = 1$, and when $b \gg 1$ all forms up to a certain number $n_* \approx \sqrt{3b}$ are also unstable. On the other hand, in the region $a \leqslant 0$, when $b \gg 1$ the lower forms are stable up to the number $n_* \approx \sqrt{3b}$, when $0 \geqslant a > -{}^1/_3$, or $n_* \approx \sqrt{b/\beta},\ \beta = 4\sqrt{3a^2 + a} - 7a - 1$, when $a \leqslant -{}^1/_3$. This difference is important in applications that will be considered below in Sects.3 and 4. Stabilization of the whole infinite set of forms $n = 0, 1, 2, \ldots$ is impossible for reasons shown above, neither in region $a \geqslant 1/b$, nor in $a \leqslant 0$. However, in the narrow band $0 < a < 1/b$ there exists a stability region of the ring in all forms. It is shown in Fig.3 shaded, and consists of separate diamond-shaped fragments joined at points C_n. Each fragment is formed at the intersection of stability regions for n and $n + 1$ between points C_n and C_{n+1}, as shown in Fig.2 for the example of $n = 5$ and $n = 6$. These fragments all lie inside the stability regions for all n.

This is shown by numerical computations and the following reasoning. Consider the curve

$$b = (1 + a)^2/(4a) \qquad (2.5)$$

which passes through all points C_n and all fragments. On curve (2.5) the left-hand side $P\,(\Omega)$ of Eq.(2.3) for $a < 0{,}0358,\ n > 0$ has the properties $P\,(\pm\infty) = +\infty,\ P\,(\pm n/\sqrt{b}) < 0,\ P\,(n\sqrt{a/b}) = 0$. It follows from this that (2.3) has four real roots for any n. At points C_n the root $\Omega = n\sqrt{a/b}$ is multiple and these points lie at the stability boundary. Non-multiple roots that are inside the stability region correspond for all n to points (2.5) that lie between the points C_n.

Outside the hatched region in Fig.3 a ring of flexible filament is, thus, unstable, while in the hatched region it is stable to a first approximation. The complete proof of stability using energy integrals and angular momentum is known for the rotating ring from an inextensible filament and when there is no external field /7/. This is a limiting case $a = \infty,\ b = 1$ relative to the parameters a and b. The presence of an attracting centre $(b > 1)$ alters the picture considerably. Proceeding as in /7/ it is not possible to obtain a complete proof of stability, since the reduced potential energy has a maximum and stabilisation has a gyroscopic character.

3. Let us now consider some applications. It was established above that in the region $a \leqslant 0$ a ring a flexible filament is stable in lower forms $n < n_*$. Instability in such a

ring only appears in higher forms $n \geqslant n_*$. The model of a ring a flexible filament is adequate for a real satellite ring in lower forms, but in higher forms does not reflect its dynamics.

Consider the following analogue of Maxwell's problem: a large number of N material points (satellites) of the same mass m_1 are successively connected in a circle by weightless cables of variable length. The satellites are numbered in the order of their connection by the numbers $k = 1, \ldots, N$. The equations of motion relative to the inertial axes $OXYZ$ have the form

$$m_1 \frac{d^2\mathbf{r}_k}{dt^2} = T_k \boldsymbol{\tau}_k - T_{k-1}\boldsymbol{\tau}_{k-1} - m_1 \mu \mathbf{r}_k r_k^{-3} \tag{3.1}$$

where $\mathbf{r}_k(t)$ is the radius vector of the k-th satellite, $\boldsymbol{\tau}_k = (\mathbf{r}_{k+1} - \mathbf{r}_k) \mid \mathbf{r}_{k+1} - \mathbf{r}_k \mid^{-1}$ is the unit vector of the direction from the k-th to the $(k+1)$-st satellite, and T_k is the cable tension between the k-th and the $(k+1)$-st satellites. Eqs.(3.1) admit of uniform rotation of the ring in a stationary plane when the satellites are arranged at the vertices of a regular N-polygon whose centre is at the point O. The tension in all cables is the same and defined by

$$T_k = T_* = m_1 \Delta_*^{-1} (\omega^2 R^2 - \mu R^{-1}), \quad \Delta_* = 2R \sin(\pi/N) \tag{3.2}$$

where R is the ring radius, ω is its angular velocity, and Δ_* is the distance between adjacent satellites.

Eqs.(3.1) and (3.2) are discrete analogues of (1.1) and (1.4), and the law of extensibility (1.3) in the discrete formulation acquires the clear meaning of the same law of tension control $T(\Delta)$ for all cables, which depends on the distance between adjacent satellites

$$T_k = T(\Delta_k), \quad \Delta_k = |\mathbf{r}_{k+1} - \mathbf{r}_k| \tag{3.3}$$

Such control can be realized if every satellite has a supply of spare cable and a device for pulling-in and releasing the cable.

Equations of small oscillations of satellites relative to the rotating axes $Oxyz$ are similar to (2.1), and the finite differences

$$\begin{aligned}
&b(u_k^{\cdot\cdot} - 2v_k^{\cdot} - 3u_k) = {}^1/_4 \operatorname{ctg}^2 \varepsilon \,(u_{k+1} - 2u_k + u_{k-1}) - \\
&\quad 2u_k - {}^1/_4 a(u_{k+1} + 2u_k + u_{k-1}) - {}^1/_4(1+a)\operatorname{ctg}\varepsilon\,(v_{k+1} - v_{k-1}) \times \\
&b(v_k^{\cdot\cdot} + 2u_k^{\cdot}) = {}^1/_4(1+a)\operatorname{ctg}\varepsilon\,(u_{k+1} - u_{k-1}) + \\
&\quad {}^1/_4 a \operatorname{ctg}^2\varepsilon\,(v_{k+1} - 2v_k + v_{k-1}) + {}^1/_4(2v_k - v_{k+1} - v_{k-1}) \\
&b(w_k^{\cdot\cdot} + w_k) = w_k + {}^1/_4 \sin^{-2}\varepsilon\,(w_{k+1} - 2w_k + w_{k-1}) \\
&\varepsilon = \frac{\pi}{N}, \quad a = \frac{\Delta_*}{T_*}\left(\frac{dT}{d\Delta}\right)_{\Delta=\Delta_*}, \quad b = \frac{m_1\omega^2R^2}{\Delta_* T_*}
\end{aligned} \tag{3.4}$$

correspond to derivatives with respect to φ. In these formulae $u_k(\tau)$, $v_k(\tau)$, $w_k(\tau)$ are the radial, transverse and axial displacements of the k-th satellite relative to its position in steady motion.

In complete analogy with (2.1) the particular solutions of (3.4) are obtained in the form $u_k = U\cos(\Omega\tau + n\varphi_k)$, $v_k = V\sin(\Omega\tau + n\varphi_k)$, $w_k = 0$ of oscillations in the Oxy plane, and in the form $u_k = v_k = 0$, $w_k = W\cos(\Omega\tau + n\varphi_k)$ for oscillations in the direction of Oz and n is an integer $|n| \leqslant N/2$, $\varphi_k = 2\pi k/N$. The characteristic equation for axial oscillations

$$b\Omega^2 = b - 1 + \sin^2(n\varepsilon)\sin^{-2}\varepsilon \tag{3.5}$$

has for all n by virtue of $b > 1$ two different real roots, which means stability in relation to axial displacements to a first approximation. The characteristic equation for oscillations in the plane of steady rotation of the ring

$$\begin{aligned}
&[b\Omega^2 + 3b - a - 2 + (a - \operatorname{ctg}^2\varepsilon)\sin^2(n\varepsilon)] \times \\
&[b\Omega^2 + (1 - a\operatorname{ctg}^2\varepsilon)\sin^2(n\varepsilon)] - \\
&[2b\Omega - (1+a)\operatorname{ctg}\varepsilon\sin(n\varepsilon)\cos(n\varepsilon)]^2 = 0
\end{aligned} \tag{3.6}$$

is the same as (2.3) strictly for $n = 0$, and accurate to small $\sim (n/N)^2$ for $n \ll N$.

The ring oscillations of connected satellites are, thus, well defined in the lower forms by the flexible ring model. However, unlike the filament ring, the ring of connected satellites has a finite number of degrees of freedom, and for it $|n| \leqslant N/2$. The higher oscillation forms in which the filament ring shows instability at $a \lessdot 0$, do not exist in a ring of connected satellites. Using the results of Sect.2, in region $a \leqslant 0$ it is necessary and sufficient to a first approximation of the ring of connnected satellites to have the highest possible form $n = N/2$ for even N and $n = (N-1)/2$ for odd N. Substituting into (3.6) for even N $n\varepsilon = \pi/2$ and $n\varepsilon = (1 - 1/N)\pi/2$ for odd N, we obtain a quadratic equation in Ω^2 strictly for even N and with an accuracy to small $\sim 1/N$ for odd N. The condition for the existence in this equation of four different real roots Ω is expressed with an accuracy to small $\sim 1/N^2$ in the form

$$b\varepsilon^2 > \beta(a) = \begin{cases} {}^1/_3, & -{}^1/_3 < a \leqslant 0 \\ 4\sqrt{3a^2+a} - 7a - 1, & a \leqslant -{}^1/_3 \end{cases} \tag{3.7}$$

or in initial notation

$$T_* < \frac{m_1\omega^2 R}{N} \frac{\pi}{2\beta(a)} \tag{3.8}$$

The rotation of a ring of connected satellite can thus be stabilized by controlling the tension of the connecting cables. The nominal tension in the cables T_* must not exceed the critical value (3.8), while in perturbed motion the tension in the cables must diminish or remain constant when the distance between two connected satellites $dT/d\Delta \leqslant 0$ increases. Maintaining the tension in the cables constant is the simplest means of stabilization.

Consider the stability region represented in Fig.3. For real values of $b \gg 1$ it has a very narrow width $\Delta a \approx 1/(2b\sqrt{3b})$, which in practice does not enable one to use the respective control laws.

4. Note the analogy between artificial and natural rings. If we take $T = Gm_1^2\Delta^{-2}$ (G is the universal gravitational constant) as the law of tension, we obtain a model of a material ring, similar to Maxwell's model, but which takes into account the gravitational interaction only between the nearest bodies. Substituting into (3.8) the values $a = -2$, $T_* = Gm_1^2(2R\sin(\pi/N))^{-2}$ that correspond to that model, we obtain (accurate to small quantities) the condition of stability

$$m < 2{,}42MN^{-2} \tag{4.1}$$

which differs from Maxwell's condition (0.1) obtained allowing for the gravitational interaction of all bodies; here $m = m_1N$ is the overall mass of the ring and M is the mass of the central body. The closeness of these results shows that in the motion of a meteoric ring the interaction between the nearest bodies is the deciding factor, and the stability is dependent on "negative elasticity" i.e. the decrease in the interaction force between the bodies of the ring as the distance between them increases. Thus, the stable rings of attached satellites are dynamically similar to meteoritic rings and are natural prototypes for arificial rings.

In turn, the investigation of artificial rings leads to an unexpected aspect of natural rings. If in the model of Sect.3 with the law (3.3), $T = Gm_1^2\Delta^{-2}$, corresponding to a meteoric ring, we pass to the limit $N \to \infty$ with $m_1N = \text{const}$, then (3.1)–(3.6) become the corresponding Eqs.(1.1)–(2.3) for the equivalent filament that obeys the law of extensibility

$$T = G\rho^2\gamma^{-2} \tag{4.2}$$

where ρ is the mass per unit of length of the ring in steady rotation, and γ is the relative elongation of a section of the ring. This analogy with filaments is completely natural when one considers the separate narrow rings in the system of rings of Saturn, Uranus, or Jupiter.

The values $a = -2$, $b \approx 2\pi M/m$ correspond to model (4.2). According to Sect.2 such a ring is stable in lower forms of oscillations of wavelength $\lambda > \lambda_* \approx R\sqrt{2\pi\beta(-2)m/M}$. For real rings /8/ the values of λ_* are less than the width of the rings ΔR. However, the model of the equivalent filament is certainly inapplicable for defining shortwave oscillations $\lambda \lesssim \Delta R$ of a meteoric ring, when it is not possible to consider as an infinitely thin filament. Hence, the model of an equivalent filament gives, within the limits of its applicability, the stability of meteoric rings, that corresponds to reality. Shortwave $\lambda \leqslant \lambda_*$ instability lies beyond the limits of applicability of the model and bears no relation to reality.

Model (4.2) can be used to describe long-wave forced oscillations of meteoric rings. Thus, the observable eccentricity and precession of rings may be considered as excitation of the form $n = 1$.

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ON THE ROTATIONAL MOTION OF A SOLID CARRYING A VISCO-ELASTIC DISC IN A CENTRAL FIELD OF FORCE*

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The motion of a mechanical system consisting of a symmetrical solid and a round plate (disc) located in the equatorial plane of the ellipsoid of inertial of the solid is considered. It is assumed that the centre of mass of the system moves in a circular orbit in a Newtonian field of force. The disc flexural deformation, accompanied by the dissipation of energy, are the cause of the development of rotational motion in the system. Approximate equations that define this development are obtained, using the method of motion separation and of averaging /1–3/. The averaged equations that define the evolution in Andoyer variables are similar to the equations that describe the evolution of motions of a satellite with flexible visco-elastic rods located along the axis of symmetry of the satellite /3/.

Let the system of equations $Cx_1x_2x_3$ be rigidly attached to a symmetric solid Cx_3 is the axis of symmetry), and let a disc be located in the plane Cx_1x_2. The radius vector of any point on the disc is defined by

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + w\mathbf{e}_3,\ \ x_1 = r\cos\theta,\ \ x_2 = r\sin\theta \qquad (1)$$
$$0 \leqslant r \leqslant a,\ \ 0 \leqslant \theta \leqslant 2\pi,\ \ (x_1,\ x_2) \in \Omega = \{x_1^2 + x_2^2 \leqslant a^2\}$$

where $w(r, \theta, t)$ is the displacement of points of the elastic disc along the axis Cx_3 during bending, $\mathbf{e}_i\ (i = 1, 2, 3)$ is the unit vector of the axis Cx_i, and r, θ are the polar coordinates in the region Ω.

Consider the problem when the centre of mass C describes around the attracting centre O a circular Keplerian orbit of radius R and the bending oscillations of the disc do not affect its motion. We introduce the system of coordinates $C\xi_1\xi_2\xi_3$, moving translationally, and the $C\xi_3$ axis is orthogonal to the plane of the orbit. The radius vector of the centre of attraction has in system $C\xi_1\xi_2\xi_3$ the projections $(R\cos\omega_0 t,\ R\sin\omega_0 t,\ 0)$, where ω_0 is the orbital angular velocity. Let μ be the gravitational constant of the Newtonian field; then $\omega_0^2 = \mu R^{-3}$.

We will henceforth assume that the description of the deformed state of the disc conforms to the usual assumptions of the linear theory of small deflections of thin plates. In particular, when considering the deflection of a disc of constant rigidity D, the potential energy functionals of elastic deformations and of dissipative forces are defined by the formulae /4/

$$E[w] = \frac{D}{2}\iint_\Omega \Big\{(\Delta w)^2 - 2(1-\nu)\Big[\frac{\partial^2 w}{\partial r^2}\Big(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}\Big) - \frac{1}{r^2}\Big(\frac{\partial^2 w}{\partial r\,\partial\theta} - \frac{1}{r}\frac{\partial w}{\partial\theta}\Big)^2\Big]\Big\} r\,dr\,d\theta \qquad (2)$$
$$D[w^{\cdot}] = \chi E[w^{\cdot}],\ \ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},\ \ D = \frac{Eh^3}{12(1-\nu^2)}$$

where Δ is the Laplace operator, D is the bending rigidity of the disc, and E, ν are the modulus of elasticity and Poisson's ratio of the material, respectively, h is the disc thickness, assumed constant, and χ is a coefficient that takes into account the dissipation of energy of deformation. The region of definition of the above functionals (2) is the Sobolev space $W_2^2(\Omega)$. The second relation in (2) assumes that the dissipative functional $D[w^{\cdot}]$ is

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